Matrix Form of the adjoint representation

$$\mathfrak{g} = \mathrm{span}\,(e_1,\ e_2,\ \ldots,\ e_n) = \mathbb{C}e_1 + \mathbb{C}e_2 + \cdots + \mathbb{C}e_n$$

$$[e_i,\ e_j] = \sum_{k=1}^{n} c_{ij}^{k}\, e_k, \qquad c_{ij}^{k} \longleftrightarrow \text{structure constants}$$

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Antisymmetry: $c_{ij}^{k} = -c_{ji}^{k}$

Jacobi identity $c_{i\,j}^{m} c_{m\,k}^{\ell} + c_{j\,k}^{m} c_{m\,i}^{\ell} + c_{k\,i}^{m} c_{m\,j}^{\ell} = 0$

$$e^{\ell} = e_{\ell}^{*} \in \mathfrak{g}^{*} \rightsquigarrow e_{\ell}^{*}.e_p = e^{\ell}.e_p = \delta_p^{\ell}$$

$$\mathbb{P}_i \in \mathfrak{gl}(\mathfrak{g}) \ : \ \boxed{e^{k}.\mathbb{P}_i e_j = (\mathbb{P}_i)_j^{k} = c_{ij}^{k}} \rightsquigarrow$$

$$[\mathbb{P}_i,\ \mathbb{P}_j] = \sum_{k=1}^{n} c_{ij}^{k}\, \mathbb{P}_k \rightsquigarrow$$

$$\boxed{\mathbb{P}_i = \mathrm{ad}_{e_i}}$$

Representations, Modules

V $\mathbb{F}$-vector space, $u, v, \ldots \in V$, $\alpha, \beta, \ldots \in \mathbb{F}$

Definition

Representation (ρ, V)

$$\mathfrak{g} \ni x \xrightarrow{\rho} \rho(x) \in \mathfrak{gl}(V)$$
$$V \ni v \xrightarrow{\rho(x)} \rho(x)v \in V$$
$$\rho(x) \text{ Lie homomorphism}$$

$$\begin{aligned} \rho(\alpha\, x + \beta\, y) = & \quad \alpha\, \rho(x) + \beta\, \rho(y) \\ \rho\,([x,\, y]) = & \quad [\rho(x),\, \rho(y)] = \\ = & \quad \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) \end{aligned}$$

$V = \mathbb{C}^n \ \rightsquigarrow\ \rho(x) \in M_n(\mathbb{C}) = \mathfrak{gl}(V) = \mathfrak{gl}\,(\mathbb{C}, n)$

$W \subset V$ invariant (stable) subspace

$\leftrightsquigarrow \rho(\mathfrak{g})W \subset W$

$\leftrightsquigarrow$ (ρ, W) is submodule $\leftrightsquigarrow (\rho, W) \prec (\rho, V)$

Το σύνολο των αναπαραστάσεων είναι μερικά διατεταγμένο.

$$\rho(x) = \begin{array}{c} \left[\begin{array}{c|c|c} Q\,\rho(x)\,Q & P\rho(x)\,Q & \\ \hline Q\,\rho(x)P = \{0\} & P\rho(x)P & \end{array}\right] \begin{array}{l} \} \in W \\ \} \; W^{\perp} \end{array} \Big\} \in V \end{array}$$

$V = W \oplus W^{\perp} \rightsquigarrow V \ni v = w + w^{\perp}, \quad w \in W, \; w^{\perp} \in W^{\perp}, \quad \rho(x) W \subset W$

επομένως $(\rho, W) < (\rho, V)$, $\quad Id = Q + P$

$$\left.\begin{array}{l} V \xrightarrow{Q} W \\ V \xrightarrow{P} W^{\perp} \end{array}\right\} \text{Projections s.t. } P^2 = P$$

$\rho(x) Q V \subset Q V \rightsquigarrow Q\rho(x) Q V \subset Q V$

$$\begin{tikzcd}
V \arrow[r, "\pi"] \arrow[d, "\rho(x)"'] & V/W \arrow[d, "\bar{\rho}(x)"] \\
V \arrow[r, "\pi"] & V/W
\end{tikzcd}$$

$$\begin{tikzcd}
V \arrow[r, "p", "epi"'] \arrow[d, "\pi"', "epi"] & W^{\perp} \arrow[dl, "\psi"] \arrow[dd, "\sigma(x)"] \\
V/W \arrow[d, "\bar{\rho}(x)"'] & \\
V/W \arrow[r, "\psi^{-1}"] & W^{\perp}
\end{tikzcd}$$

$\sigma(x) = \psi^{-1} \circ \bar{\rho}(x) \circ \psi$

$\sigma([x,y]) = [\sigma(x), \sigma(y)]$

Επομένως

$(\sigma, W^{\perp}) \underset{iso}{\cong} (\bar{\rho}, V/W)$

ΠΡΟΣΟΧΗ

$\psi \in Aut_{Lin}(W^{\perp}, V/W)$

δηλ. $\exists \psi^{-1}$.

Θέτουμε $\sigma(x) = \psi^{-1} \circ \rho(x) \circ \psi$

Τότε $\sigma([x,y]) = \sigma(x) \circ \sigma(y) - \sigma(y) \circ \sigma(x) =$

$= \psi^{-1} \bar{\rho}(x) \circ \underbrace{\psi \circ \psi^{-1}}_{Id} \bar{\rho}(y) \psi - \psi^{-1} \circ \rho(x) \circ \underbrace{\psi \circ \psi^{-1}}_{Id} \rho(y) \psi =$

$= \psi^{-1} \bar{\rho}([x,y]) \circ \psi .$

Το $(\sigma, W^{\perp})$ είναι αναπαράσταση και $(\sigma, W^{\perp}) \underset{is}{\cong} (\bar{\rho}, V/W)$

Theorem

W invariant subspace of V $\leftrightsquigarrow (\rho,\ W) \prec (\rho,\ V)$

$(\rho, V) \rightsquigarrow \ \exists!$ $\boxed{\textit{induced representation}}$ $(\overline{\rho}, V/W)$ (quotient module)

$$\rho(\mathfrak{g})W \subset W \implies \exists!\,\overline{\rho}(x) :\ V/W \longrightarrow V/W$$

ΣΗΜΕΙΩΣΗ 13

$$\implies \begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\rho(x)"'] \arrow[dr, dotted] & V/W \arrow[d, dashed, "\overline{\rho}(x)"] \\ V \arrow[r, "\pi"] & V/W \end{tikzcd} \quad \leftrightsquigarrow \quad \overline{\rho}(x) \circ \pi = \pi \circ \rho(x)$$

$\mathrm{Ker}\,\rho = C_\rho(\mathfrak{g}) = \{x \in \mathfrak{g}\ :\ \rho(x)V = \{0\}\}$ is an ideal

$\mathrm{Ker}\,\mathrm{ad} = C_{\mathrm{ad}}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) =$ center of $\mathfrak{g}$

Αν $\rho = \mathrm{ad}$

$\rightsquigarrow C_{\mathrm{ad}}(\mathfrak{g}) = Z(\mathfrak{g})$

διότι $\mathrm{ad}_{C_{\mathrm{ad}}} = 0$.

$$\mathfrak{g} \xrightarrow[\text{Lie-epi}]{\rho} \rho(\mathfrak{g})$$

$x \in \mathrm{Ker}\,\rho \rightsquigarrow \rho(x)V = \{0\} \rightsquigarrow x \in C_\rho(\mathfrak{g})$

$x \in C_\rho(\mathfrak{g}) \rightsquigarrow \rho(x)V = \{0\} \rightsquigarrow x \in \mathrm{Ker}\,\rho$.

δηλ. $\boxed{\rho(C_\rho) = 0}$

ΣΗΜΕΙΩΣΗ 14

Αν $(\rho, W) \prec (\rho, V)$ τότε υπάρχει $(\bar{\rho}, V/W)$ έτσι ώστε $\forall x \in G$

το διάγραμμα είναι κλειστό

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\rho(x)"'] & V/W \arrow[d, "\bar{\rho}(x)"] \\ V \arrow[r, "\pi"] & V/W \end{tikzcd}$$

δηλ $\bar{\rho}(x) \circ \pi = \pi \circ \rho(x)$ $\quad (*)$

Απόδειξη.

$$\begin{tikzcd} V \arrow[d, "\rho(x)"'] \arrow[dr, "\pi\circ\rho(x)"] & \\ V \arrow[r, "\pi"', "epi"] & V/W \end{tikzcd}$$

Αν $v \in W \rightsquigarrow \rho(x)v \in W \rightsquigarrow \pi(\rho(x)v) = 0 \rightsquigarrow$
$v \in \operatorname{Ker}(\pi \circ \rho(x)) \rightsquigarrow W \subseteq \operatorname{Ker}(\pi \circ \rho(x))$

$$\begin{tikzcd} V \arrow[r, "\pi", "epi"'] \arrow[dr, "\pi\circ\rho(x)"'] & V/W \arrow[d, "!\,\bar{\rho}(x)"] \\ & V/W \end{tikzcd}$$

$$\left.\begin{array}{l} \operatorname{Ker}\pi = W \subseteq \operatorname{Ker}(\pi \circ \rho) \\ \pi \text{ epi} \end{array}\right\} \Rightarrow \exists!\, \bar{\rho}(x)$$

και το διάγραμμα $(*)$ είναι κλειστό

δηλ $\bar{\rho}(x) \circ \pi = \pi \circ \rho(x)$

Representations on Hommomorhism Modules

ΕΡΓΑΣΙΑ 2

Let (ρ_1, V_1) and (ρ_1, V_1) $\mathfrak{g}$-representations

Definition: Homomorphism module

An Homomorphism Module is the pair $(\tau, \mathrm{Hom}_{\mathbb{C}}(\mathrm{V}_1, \mathrm{V}_2))$, where

$$\mathfrak{g} \ni x \xrightarrow{\tau} \tau(x) \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(\mathrm{V}_1, \mathrm{V}_2), \mathrm{Hom}_{\mathbb{C}}(\mathrm{V}_1, \mathrm{V}_2))$$

and if $f \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{V}_1, \mathrm{V}_2)$

$$\tau(x)(f) = \rho_2(x) \circ f - f \circ \rho_1(x)$$

We can prove

$$\tau([x, y]) = \tau(x) \circ \tau(y) - \tau(y) \circ \tau(x)$$

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Two representations are equivalent if there is an $f \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{V})$ such that $\tau(f) = 0$

Reducible and Irreducible representation

(ρ, V) irreducible/simple representation (irrep)

$\leftrightsquigarrow$ $\nexists$ (non trivial) invariant subspaces

$\leftrightsquigarrow$ $\rho(\mathfrak{g})W \subset W \;\Rightarrow\; W = \{0\}$ or V

$\leftrightsquigarrow$ $\boxed{(\rho, W) \prec (\rho, V) \;\Rightarrow\; W = \{0\} \text{ or } V}$

ΣΗΜΕΙΩΣΗ 15

(ρ, V) reducible/semisimple representation

$\leftrightsquigarrow$ $\boxed{V = V_1 \oplus V_2}$, $(\rho,\ V_1)$ and $(\rho,\ V_2)$ submodules

$V = V_1 \oplus V_2 \rightsquigarrow\ \rho(\mathfrak{g})V_1 \subset V_1, \quad \rho(\mathfrak{g})V_2 \subset V_2$

(ρ, V) trivial representation

$\leftrightsquigarrow$ $V = \bigoplus_{\ell=1}^{k} V_\ell$ where $V_\ell = \mathbb{F}$ i.e $\dim V_\ell = 1$

reducible representation

$$\rho(x) = \begin{pmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{pmatrix} \qquad \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \begin{matrix} \left.\vphantom{\vdots}\right\} V_1 \\ \left.\vphantom{\vdots}\right\} V_2 \end{matrix}$$

Jordan Hölder decomposition

$\boxed{(\rho, W) \prec (\rho, V) \text{ and } \boxed{(\overline{\rho}, V/W) \text{ \underline{not} irrep \underline{nor} trivial}}}$

$\rightsquigarrow \exists\ (\overline{\rho}, \overline{U}) \prec (\overline{\rho}, V/W) \rightsquigarrow W \subset U = \pi^{-1}(\overline{U}) \subset V$

$$\rightsquigarrow \quad \begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\rho(x)"'] \arrow[dr, dotted] & V/W \arrow[d, dashed, "\overline{\rho}(x)"] \\ V \arrow[r, "\pi"] & V/W \end{tikzcd} \quad \leftrightsquigarrow \quad \overline{\rho}(x) \circ \pi = \pi \circ \rho(x)$$

$$(\pi \circ \rho(x))\,(U) = (\overline{\rho}(x) \circ \pi)\,(U) = \overline{\rho}(x)\,(\overline{U}) \subset \overline{U} = \pi(U) \rightsquigarrow \ \rho(x)(U) \subset U$$

$$\rightsquigarrow \boxed{\boxed{(\rho, W) \prec (\rho, U) \prec (\rho, V)}}$$

ΣΗΜΕΙΩΣΗ 16

Lemma

For every representation (ρ, V), which is not irrep or trivial, there is a subrepresentation $(\rho, U) \prec (\rho, V)$ such that $(\overline{\rho}, V/U)$ is an irrep or a trivial representation.

ΣΗΜΕΙΩΣΗ 17

Theorem (Jordan Hölder decomposition)

(ρ, V) representation, then
$V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_m = \{0\}$, $(\rho, V_i) \succ (\rho, V_{i+1})$ *and*
$\left(\overline{\rho}_{V_i/V_{i+1}}, V_i/V_{i+1}\right)$ *irrep or trivial*

ΣΗΜΕΙΩΣΗ 18

ΣΗΜΕΙΩΣΗ 16 $\boxed{(\rho,W)<(\rho,V) \text{ and } (\bar\rho,V/W) \text{ not irrep or trivial}}$ $\Rightarrow$ $\boxed{\exists\, U \text{ τ.ω. } W\subset U\subset V,\ (\rho,W)<(\rho,U)<(\rho,V)}$

Απόδειξη: $(\bar\rho, V/W)$ not irrep $\Leftrightarrow$ $\exists\ \bar U=\pi(U)\neq \bar 0=\pi(W)$ and $(\bar\rho,\bar U)<(\bar\rho,V/W) \rightsquigarrow W<U<V$

$$\begin{array}{ccc} V & \xrightarrow[\text{epi}]{\pi} & V/W \\ \text{ξέρουμε ότι } \rho(x)\downarrow & & \downarrow \bar\rho(x) \\ V & \xrightarrow[\text{epi}]{\pi} & V/W \end{array} \rightsquigarrow \pi\circ\rho(x)=\bar\rho(x)\circ\pi.$$

Έστω $\bar u\in\bar U \rightsquigarrow (\pi\circ\rho(x))u=(\bar\rho(x)\circ\pi)u=\bar\rho(x)(\overbrace{\pi(u)}^{\bar u\in\bar U})=\bar\rho(x)\bar u\in\bar U=\pi(U)\rightsquigarrow$

$\rightsquigarrow (\pi\circ\rho(x))u=\pi(\rho(x)u)\in\pi(U)$ άρα $\rho(x)u\in U\rightsquigarrow$

$\rightsquigarrow (\rho,U)$ is a representation and $(\rho,W)<(\rho,U)<(\rho,V)$

ΣΗΜΕΙΩΣΗ 17 $\boxed{\exists\,(\rho,U)<(\rho,V) \text{ τ.ω. } (\bar\rho,V/U) \text{ irreducible or trivial}}$

Απόδειξη: (ρ,V) όχι irrep $\rightsquigarrow \exists\,(\rho,U)<(\rho,V)$ αν $(\bar\rho,V/U)$ irrep or trivial τελειώσαμε αν όχι υπάρχει τότε υπάρχει U_1 τέτοιο ώστε $(\rho,U)<(\rho,U_1)<(\rho,V)$. Αν $(\bar\rho,V/U_1)$ irrep or trivial τελειώσαμε αν όχι. τότε υπάρχει U_2 τ.ω. $(\rho, U_1) \triangleleft (\rho, U_2) \triangleleft (\rho, V)$. Και συνεχίζουμε με τον ίδιο τρόπο.

Επειδή $\dim V < \infty$ τότε υπάρχει κάποιο U' έτσι ώστε $(\rho, U') \triangleleft (\rho, V)$ και $(\bar{\rho}, V/U')$ irrep or trivial.

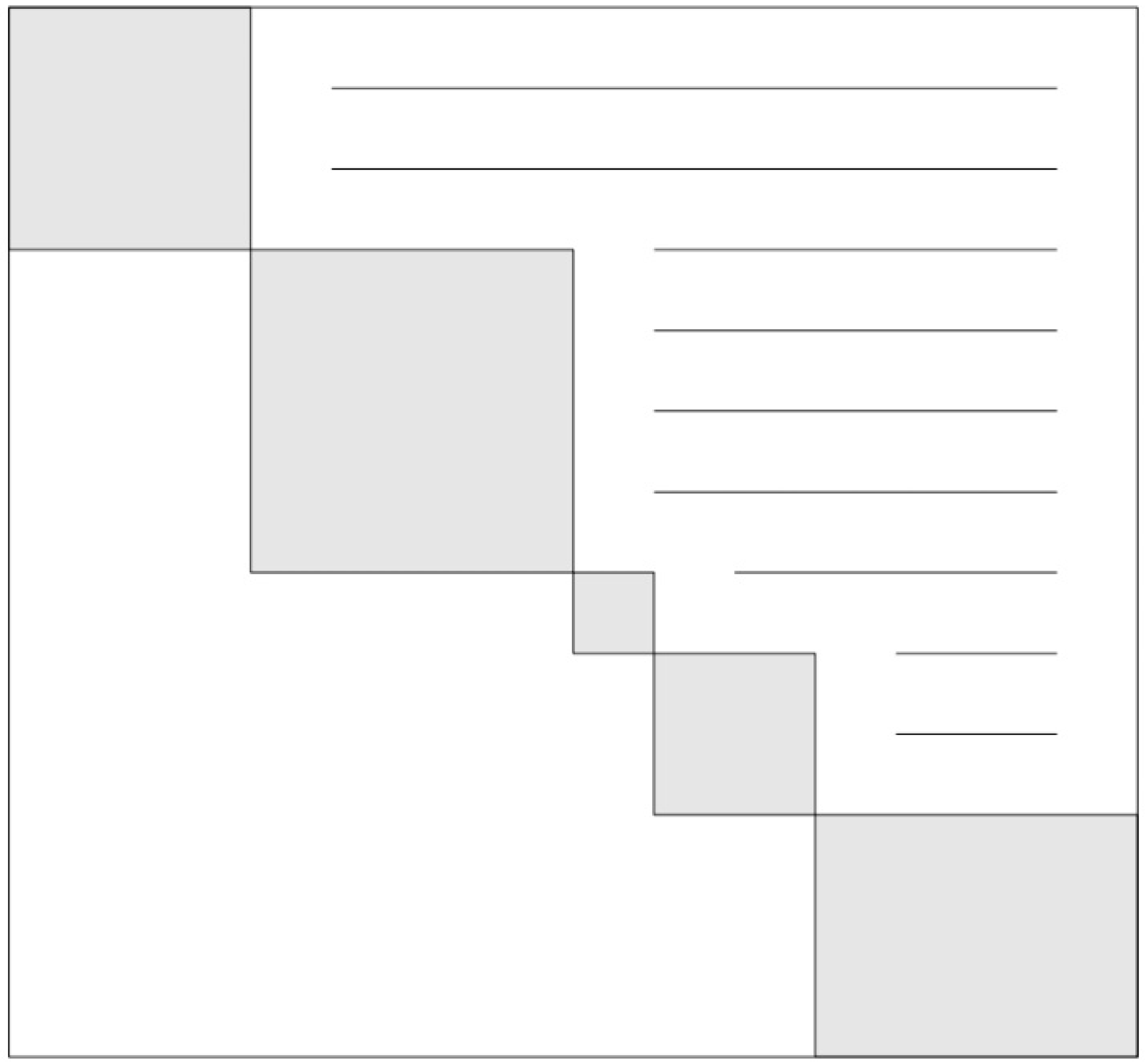
ΣΗΜΕΙΩΣΗ 18

Θεώρημα Jordan - Hölder. Αν (ρ, V) not irrep or trivial. Τότε υπάρχουν sub-modules (ρ, V_i) τέτοια ώστε:

$$(\rho, V_0) \succ (\rho, V_1) \succ (\rho, V_2) \succ \cdots \succ (\rho, V_m)$$

Με $V_0 = V$ και $V_m = \{0\}$ και $(\bar{\rho}_i, V_i / V_{i+1})$ irrep or trivial.

Η απόδειξη είναι άμεση.

$\rho(x) =$ 

Jordan Hölder Theorem σημαίνει ότι υπάρχει μία βάση στο V τέτοια ώστε για ΟΛΑ τα $x \in G$ το $\rho(x)$ είναι της παραπάνω μορφής.

Direct sum of representations

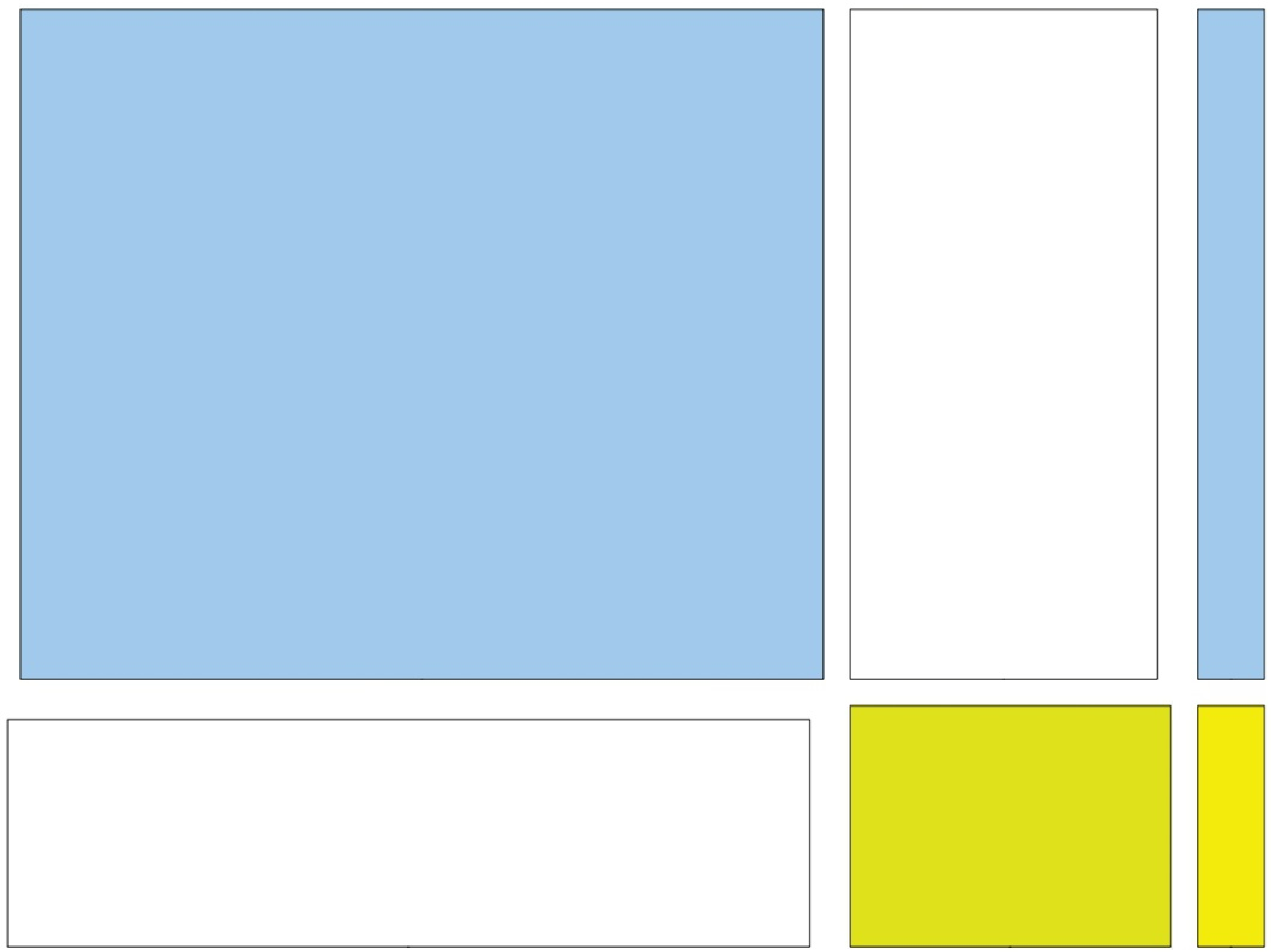
(ρ_1, V_1), (ρ_2, V_2) representations of $\mathfrak{g}$

Def: **Direct sum of representations** $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$

$$(\rho_1 \oplus \rho_2)(x) \ : \ V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$$

$$V_1 \oplus V_2 \ni (v_1,\, v_2) \longrightarrow (\rho_1(x)v_1,\, \rho_2(x)v_2) \in V_1 \oplus V_2$$

$$V_1 \oplus V_2 \ni v_1 + v_2 \longrightarrow \rho_1(x)v_1 + \rho_2(x)v_2 \in V_1 \oplus V_2$$



Theorem

Jordan- Hölder $\rightsquigarrow$ *Any representation of a simple Lie algebra is a direct sum of simple representations or trivial representations*

Θεώρημα Weyl.

(Θα συζητήσουμε αργότερα το θεώρημα Weyl)